Mathematical Modeling

Mathematical modeling is an important component of the kind of thinking done by engineers.

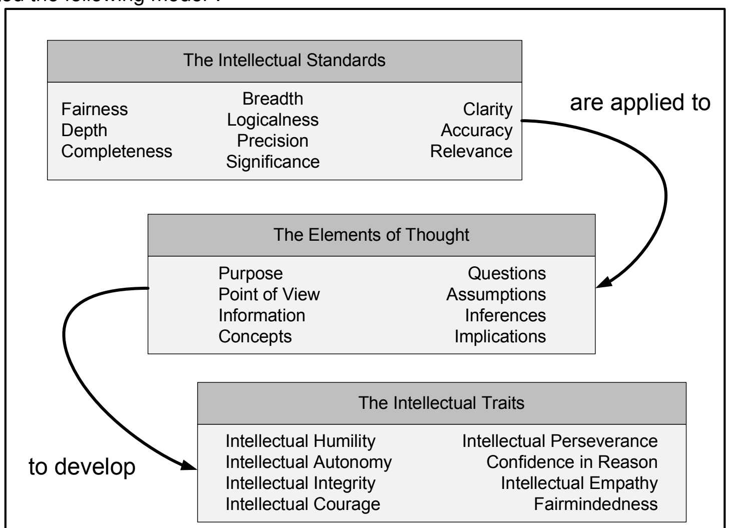
We can understand engineering thinking (or reasoning) as discipline specific critical thinking.

What is Engineering Reasoning or Critical thinking?

The University has developed the following definition:

> *Critical thinking is the intellectually disciplined process of actively and skillfully conceptualizing, applying, analyzing, synthesizing, and/or evaluating information gathered from, or generated by, observation, experience, reflection, reasoning, or communication,* ***as a guide to belief or action***.[1]

and adopted the following model[2]:



Paul-Elder Model of Engineering Reasoning or Critical Thinking

The following pages contain your text's comments on Mathematical Modeling [3]. While reading these few paragraphs, underline or highlight as many words (or phrases or terms you think have very similar meaning) that coincide with the intellectual standards and the elements of thought from the Paul-Elder model.

[1] http://louisville.edu/ideastoaction/what/critical-thinking/what-is-critical-thinking/what-is-critical-thinking.html
[2] Paul, R., Niewoehner, R., & Elder, L. (2006). The Thinker's Guide to Engieering Reasonsing. The Foundation for Critical Thinking.
[3] Nagle, K.R., Saff, E. B., and Snider, A.D., 2008, *Fundamentals of Differential Equations*, 7th ed. (Boston: Pearson/Addison Wesley)

Mathematical Models

Adopting the Babylonian practices of careful measurement and detailed observations, the ancient Greeks sought to comprehend nature by logical analysis. Aristotle's convincing arguments that the world was not flat, but spherical, led the intellectuals of that day to ponder the question: What is the circumference of Earth? And it was astonishing that Eratosthenes managed to obtain a fairly accurate answer to this problem without having to set foot beyond the ancient city of Alexandria. His method involved certain assumptions and simplifications: Earth is a perfect sphere, the sun's rays travel parallel paths, the city of Syene was 5000 stadia due south of Alexandria, and so on. With these idealizations, Eratosthenes created a mathematical context in which the principles of geometry could be applied.

Today, as scientists seek to further our understanding of nature and as engineers seek, on a more pragmatic level, to find answers to technical problems, the technique of representing our "real world" in mathematical terms has become an invaluable tool. This process of mimicking reality by using the language of mathematics is known as **mathematical modeling**.

Formulating problems in mathematical terms has several benefits. First, it requires that we clearly state our premises. Real-world problems are often complex, involving several different and possibly interrelated processes. Before mathematical treatment can proceed, one must determine which variables are significant and which can be ignored. Often, for the relevant variables, relationships are postulated in the form of laws, formulas, theories, etc. These assumptions constitute the **idealizations** of the model.

Mathematics contains a wealth of theorems and techniques for making logical deductions and manipulating equations. Hence it provides a context in which analysis can take place free of any preconceived notions of the outcome. It is also of great practical importance that mathematics provides a format for obtaining numerical answers via a computer.

The process of building an effective mathematical model takes skill, imagination, and objective evaluation. Certainly an exposure to several existing models that illustrate various aspects of modeling can lead to a better feel for the process. Several excellent books and articles are devoted exclusively to the subject.

Formulate the Problem

Here you must pose the problem in such a way that it can be "answered" mathematically. This requires an understanding of the problem area as well as the mathematics. At this stage, you may need to spend time talking with nonmathematicians and reading the relevant literature.

Develop the Model

There are two things to be done here. First you must decide which variables are important and which are not. The former are then classified as independent variables or dependent variables. The unimportant variables are those that have very little or no effect on the process. (For example, in studying the motion of a falling body, its color is usually of little interest.) The independent variables are those whose effect is significant and that will serve as input for the model. For the falling body, its shape, mass, initial position, initial velocity, and time from release are possible independent variables. The dependent variables are those that are affected by the independent variables and that are important to solving the problem. Again, for a falling body, its velocity, location, and time of impact are all possible dependent variables.

Second, you must determine or specify the relationships (for example, a differential equation) that exist among the relevant variables. This requires a good background in the area and insight into the problem. You may begin with a crude model and then, based upon testing, refine the model as needed. For example, you might begin by ignoring any friction acting on the falling body. Then, if necessary to obtain a more acceptable answer, try to take into account any frictional forces that may affect the motion.

Test the Model

Before attempting to "verify" a model by comparing its output with experimental data, the following questions should be considered:

Are the assumptions reasonable?

Are the equations dimensionally consistent? (For example, we don't want to add units of force to units of velocity.)

Is the model internally consistent in the sense that equations do not contradict one another?

Do the relevant equations have solutions?

Are the solutions unique?

How difficult is it to obtain the solutions?

Do the solutions provide an answer for the problem being studied?

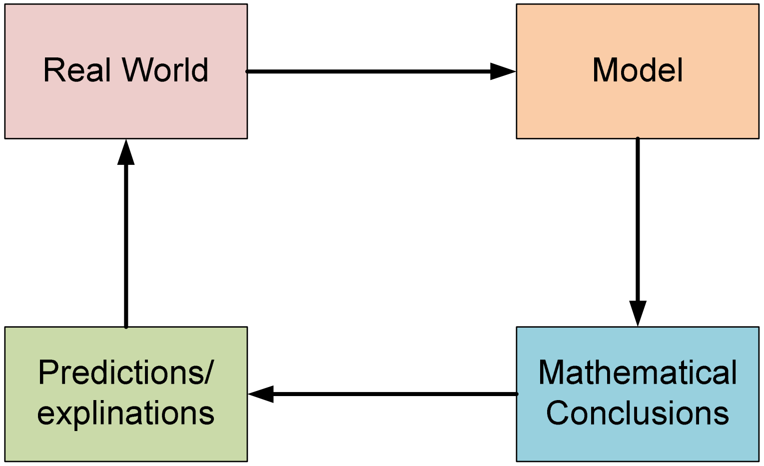
When possible, try to validate the model by comparing its predictions with any experimental data. Begin with rather simple predictions that involve little computation or analysis. Then, as the model is refined, check to see that the accuracy of the model's predictions is acceptable to you. In some cases, validation is impossible or socially, politically, economically, or morally unreasonable. For example, how does one validate a model that predicts when our sun will die out?

Each time the model is used to predict the outcome of a process and hence solve a problem, it provides a test of the model that may lead to further refinements or simplifications. In many cases, a model is simplified to give a quicker or less expensive answer—provided, of course, that sufficient accuracy is maintained.

One should always keep in mind that a model is *not* reality but only a representation of reality. The more refined models *may* provide an understanding of the underlying processes of nature. For this reason, applied mathematicians strive for better, more refined models. Still, the real test of a model is its ability to find an acceptable answer for the posed problem.

Engineering Critical Thinking: Critical Thinking in Differential Equations

On the diagram[4] below use **Blue Ink** (engineering students have Tablet PCs) to label the boxes and arrows with elements thought from the Paul-Elder model. Be prepared to justify you labeling. You may use elements in more than one place and you may label boxes and arrows with more than one element



Now, using **Red Ink**, do the same thing using the intellectual standards from the Paul-Elder model.

[4] Adapted from: Thomas, G.B, Weir, M. D., Hass, J. and Giordano, F.R., 2005, Thomas' Calculus, 11/E (Boston: Pearson/Addison Wesley).

Critical Thinking Assignment #1

For the two commentaries that follow, which one better exemplifies good critical thinking? Provide a brief justification for your answer (between one-half and one page, type written).

Commentary #1[5]

The U.S. government will soon have no choice but to make birth control mandatory. From the data in table titled US Census Data it is clear that the population in the US has drastically increased since the census was first taken in 1800. Current growth is a predictor of how the population will grow in the future. For each census count the increase is greater than the last time, therefore the change in the population is proportional to the population size; the larger the population the greater the increase in the population size. Therefore equation (1) can be used to model how the population is changing.

$$\frac{dp}{dt} = kp \; (1)$$

Equation (2) is a general solution to (1). Data from the table can be used to find the particular solution that represents the U.S. population.

$$p(t) = p_0 e^{kt} \;\; (2)$$

US Census Data

Year	U.S. Census count in million
1800	5.31
1810	7.24
1820	9.64
1830	12.87
1840	17.07
1850	23.19
1860	31.44
1870	39.82
1880	50.19
1890	62.98
1900	76.21
1910	92.23
1920	106.02
1930	123.20
1940	132.16
1950	151.33
1960	179.32
1970	203.30
1980	226.54
1990	248.71

According to the census data, the initial population count was 5.31 million. Letting $k = 0.026639$, equation (2) becomes equation (3).

$$p(t) = 5.31e^{0.026639t} \;\; (3)$$

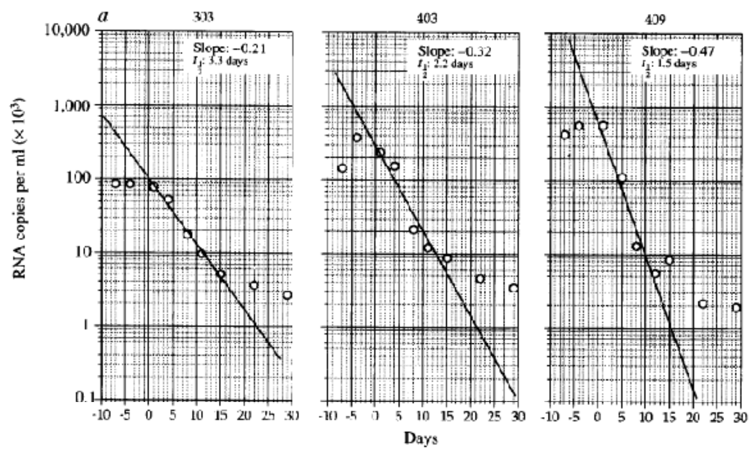
This model accurately predicts the population in 1900 to be 76.21 million. Equation 3 proves that the U.S. population is not just increasing but is increasing exponentially without bound. By 2020 there will be more than 1.8 billion people in the US. There is not enough room in the United states for this many people. When there are too many people and not enough resources, civil unrest will breakout, and possibly even anarchy. The future is not bright for the United States of America.

[5] Some of the information in this commentary, including the table, is drawn from the text book: Nagle, K.R., Saff, E. B., and Snider, A.D., 2008, *Fundamentals of Differential Equations*, 7th ed. (Boston: Pearson/Addison Wesley)

Engineering Critical Thinking: Critical Thinking in Differential Equations

Commentary #2[6]

Experience with HIV indicates that the disease typically exhibits a lengthy gradual progression lasting many years, and the lifespan of infected CD4+ T cells is long, on the order of years. However, the viral load decrease for patents, shown in the above figure, when compared to a model of HIV infection dynamics, suggests that in fact the lifespan of infected cells is actually quite short, on the order of 2 to 5 days. Within an infected person the HIV virus spends part of its existence free and part inside an infected CD4+ T cell. The time spent "free" is known to be very short, on the order of 30 minutes. Eventually the infected CD4+ T cells burst (and is destroyed) releasing multiple new virus particles. By comparing the observed decrease in viral load of patients receiving treatment with model predictions of best case viral load decrease it can be shown that the turnover rate for the infected CD4 lymphocytes in HIV infection is on the order of about 2 to 5 days.



The dynamics of HIV infection can be modeled using compartmental analysis, with three compartments:

$T(t)$ – the population of uninfected CD4+ T cells at time t
$I(t)$ -- the population of infected CD4+ T cells at time t
$V(t)$ -- the population of virus at time t
using the following model parameters:
λ -- rate, in cells per day, that the average human body produces (uninfected) CD4+ T cells

[6] This commentary is adapted from material in the text book: Nagle, K.R., Saff, E. B., and Snider, A.D., 2008, *Fundamentals of Differential Equations*, 7th ed. (Boston: Pearson/Addison Wesley), and is supplemented from Wei. et al, 1995, "Viral dynamics in human immunodeficiency virus type 1 infection", Nature, vol 373, issue 12, January 1995, pp. 117-122

δ -- the normal loss rate, in cells per day, of CD4+ T cells due to normal cell death
β -- the infection rate constant of uninfected cells per infected cell
μ -- the loss rate constant, per day, of infected cells ($1/_\mu$ *is the average lifespan of an infected cell*)
γ -- the loss rate of free virus, per day, $\sim {}^{48}/_{day}$, $1/_\gamma$ the average life span of a free virus (or virion) in days.
N -- the number of virions produced per day per infected cell (number of virions expelled when a single infected CD4+ T cell bursts.

Compartmental analysis produces the following differential equations which model the dynamics of HIV infection.
$\frac{dT}{dt} = \lambda - \delta T(t) - \beta V(t)T(t)$; $\frac{dI}{dt} = \beta V(t)T(t) - \mu I(t)$; $\frac{dV}{dt} = N\mu I(t) - \gamma V(t)$

Assuming that a treatment is 100% effective, then $\beta = 0$ and the differential equations can be reduced. Solving each differential equation using $T(0) = T_0, I(0) = I_0, V(0) = V_0$ leads to an equation for the population of the virus at time t or the viral load at time t: $V(t) = \left(\frac{N\mu I_0 e^{t(-\mu+\gamma)}}{-\mu+\gamma} + V_0 - \frac{N\mu I_0}{-\mu+\gamma}\right)e^{-\gamma t}$. Over an extended period of time, many weeks for example, a graph of the log of $V(t)$ versus time will approach a straight line, whose slope is either $-\gamma$ or $-\mu$, whichever one is smaller. The logarithmic graph of observed viral loads, shown in the above figure, should agree with the models prediction of the graph of the natural log of $V(t)$. The figure shows a regression line for the observed viral loads in the three cases, and a slope for the regression line in each case is shown on the figure. If $\gamma \cong 48$ then μ must be the dominant term, since $\gamma \cong 48$ does not agree with the observed slopes. If the average lifespan of an infected CD4+ T cell is on the order of years, then μ is very small indeed, in fact too small to agree with observations. Instead it must be the case μ is roughly between 0.2 and 0.5, which means the average lifespan of an infected cell, $1/_\mu$, is between 2.5 days and 5 days, much shorter than originally though.

Since the lifespan of infected cells is short, the dynamics of HIV infection are characterized by a high turnover rate of the virus. Therefore, if a newly developed antiviral is effective at stopping new infections this will best be observed in the few immediate days following treatment. Otherwise, the possibility of a new strain developing and masking the effectiveness of the antiviral is significantly increased.

Engineering Critical Thinking: Critical Thinking in Differential Equations

Critical Thinking Assignment #2

During this course we have discussed different methods for solving different types of differential equations. Pick two of those methods and compare their importance. Your response should be no longer than one page and should be type written. The following rubric will be used in evaluating your response.

Consistently does all or most of the following:

4	Clearly identifies the purpose including all complexities of relevant questions. Accurate, complete information that is supported by relevant evidence. Complete, fair presentation of all relevant assumptions and points of view. Clearly articulates significant, logical implications and consequences based on relevant evidence.
3	Clearly identifies the purpose including some complexities of relevant questions. Accurate, mostly complete information that is supported by evidence. Complete, fair presentation of some relevant assumptions and points of view. Clearly articulates some implications and consequences based on evidence.
2	Identifies the purpose including irrelevant and/or insufficient questions. Accurate but incomplete information that is not supported by evidence. Simplistic presentation that ignores relevant assumptions and points of view. Articulates insignificant or illogical implications and consequences that are not supported by evidence.
1	Unclear purpose that does not includes questions. Inaccurate, incomplete information that is not supported by evidence. Incomplete presentation that ignores relevant assumptions and points of view. Fails to recognize or generates invalid implications and consequences based on irrelevant evidence.